

FINITE DIMENSIONAL APPROXIMATIONS OF THE OPERATOR EQUATIONS

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ABSTRACT. The problem of finding solution of a tridiagonal operator equation through its finite dimensional truncations is discussed. Effectively verifiable sufficient conditions are given. An algorithm is presented to compute the numerical approximation to the solution of $Tx = y$ for a given tridiagonal operator. This is illustrated with a numerical example.

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1. Introduction

Let H be a separable Hilbert space with an inner product \langle, \rangle and an orthonormal basis $E = \{e_n : n = 1, 2, 3, \dots\}$. Let H_n denote the linear span of $\{e_1, \dots, e_n\}$, P_n , the orthogonal projection of H onto H_n . For $x = \sum_i \alpha_i e_i \in H$, we shall denote by $x_n = P_n(x) = \sum_{i=1}^n \alpha_i e_i \in H_n$. Let $T \in B(H)$, the class of bounded linear operators on H and let $T_n = P_n T|_{H_n}$. T_n will be called truncation of T to H_n . These are also known as Galerkin approximations or finite sections[5]. The spectrum of T will be denoted by $\sigma(T)$. Throughout this note we shall not distinguish between the operator T and its matrix $[\langle T e_j, e_i \rangle]$ with respect to the basis E . Thus matrix of T_n will consist of the first n rows and n columns of the matrix of T .

There are two very natural questions associated with this setting:

- (1) Can we compute $\sigma(T)$ (or at least find some information about $\sigma(T)$) if we know $\sigma(T_n)$ for all n ?
- (2) Given $y \in H$, can we derive some information about solution(s) of the operator equation $Tx = y$ if we know solutions of the equations $T_n x^n = y_n$ for all n ?

Regarding the first question, it is well known that attempts to derive some information about $\sigma(T_n)$ from $\sigma(T)$ can fail dramatically. A well known example in this connection is that of the right shift operator $T : l^2 \rightarrow l^2$ defined by

$$T(x) = (0, \alpha_1, \alpha_2, \dots), \quad x = (\alpha_1, \alpha_2, \dots) \in l^2.$$

Then $\sigma(T_n) = \{0\}$ for all n , where as $\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}$ (cf. Arveson [1]). Similar situation holds for the second question as well.

Even if all finite dimensional truncations T_n of an operator T are invertible, T may not be invertible. For example, consider $T : l^2 \rightarrow l^2$ defined by

$$Tx = \left(\alpha_1, \frac{\alpha_2}{2}, \dots, \frac{\alpha_n}{n}, \dots \right), \quad x = (\alpha_1, \alpha_2, \dots) \in l^2$$

Then each T_n is invertible and T_n^{-1} is given by $T_n^{-1}(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, 2\alpha_2, \dots, n\alpha_n)$. But T is not invertible as $\sigma(T) = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

This raises a natural question: Are there any conditions that the operator T must satisfy so that these attempts have some success?

Arveson ([1],[2]) has given the following answer in the context of the first question: the success of these attempts (of computing $\sigma(T)$ from $\sigma(T_n)$) depends on whether T belongs to a certain C^* algebra. In view of this, Arveson suggests the following:

”Numerical problems involving infinite dimensional operators require reformulation in terms of C^* algebras.”

Similar issues about the second question are discussed by Bottcher and Silbermann [5] for Toeplitz and Block Toeplitz operators. In this note, we deal with this question for a tridiagonal operator. The next section contains a theorem (Theorem 2.1) which gives sufficient conditions for the convergence of $T_n^{-1}(y_n)$ to $T^{-1}(y)$. We also give effectively verifiable criterion to check that the condition in this theorem are satisfied (Corollary 2.2). In the last section, we present an algorithm to implement this procedure and illustrate this with a numerical example. This algorithm is presented only for the purpose of illustration of the main results. The questions of speed, storage, complexity, stability, possible intermediate break-down etc. of this algorithm are not yet investigated and hence no claims are made in this regard.

The above example also highlights the essential difference in operators defined on finite dimensional and infinite dimensional Hilbert spaces. We know that in the case of finite matrices, row or column dominance will lead to the invertibility. The above example shows that this is not true in the case of infinite matrices. However if there is a strict row dominance as well as column dominance one can show that the operator is invertible (see [4]). Then the natural question is whether we can extend the notion to a block of rows and columns satisfying the dominance property, in which each row and column need not satisfy such a property. In fact, in [4], it is shown that if all finite truncations T_n of a tridiagonal operator T have in appropriate sense row and column dominance property, then T satisfies the hypothesis of Theorem 2.1. of this note. (see also Corollary 2.2 and example 2.3).

2. Tridiagonal Operators

If an operator can be reduced to a diagonal form, then it is easy to answer the above questions. But reducing the operator into a diagonal form is rather exceptional and a more realistic approach is to find a basis in which the operator has a sparse representation. A more

common example of a sparse matrix is a band limited matrix, with a finite band surrounding the main diagonal. The simplest among the band limited matrices is a tridiagonal operator. The tridiagonal operators themselves find their importance in various problems such as boundary value problems with finite difference method, cubic spline interpolation, almost Mathieu operators discretized Schrödinger operators, stochastic models and so on.

In the case of tridiagonal operators, we have the following result for a certain class of operators. These operators, called almost Mathieu operators which are defined by

$$Te_n = e_{n-1} + \lambda \cos(2n\pi\alpha + \theta)e_n + e_{n+1}, \quad \alpha, \lambda, \theta \in \mathbb{R}.$$

The authors in [3] proved that if $\lambda \in \mathbb{R}$ and r is a rational multiple of π , then

$$Te_n = e_{n-1} + \lambda \sin(2nr)e_n + e_{n+1}$$

is not invertible. However, when r is an irrational multiple of π , the problem of invertibility still remains open.

If T is a general tridiagonal operator, the following result is proved in [4].

Theorem 2.1. Let T be the tridiagonal operator defined by

$$Te_n = c_{n-1}e_{n-1} + d_n e_n + u_{n+1}e_{n+1}$$

where $\{c_n\}, \{d_n\}$ and $\{u_n\}$ are bounded sequences of complex numbers. Suppose T_n is invertible for all n and there exists a constant K such that $0 < K < \infty$ and $\|T_n^{-1}e_n\| \leq K$ for all n . If the operator equation $Tx = y$ has a solution (i.e. if $y \in R(T)$, the range of T), then this solution can be obtained as the limit of the solutions x^n of the operator equation $T_n x^n = y_n|_{H_n}$ in the norm topology. In other words, $T_n^{-1}(y_n) \rightarrow x$. In particular, T is 1-1.

If, in addition $c_n \neq 0, u_n \neq 0$ for all n and $\exists L > 0$ such that $\|T_n^{*-1}(e_n)\| \leq L$ for all n then T is onto and hence invertible.

We indicate the outline of the proof for the sake of completeness. The tridiagonal nature of T implies that $T_n(x_n)$ differs from $T(x_n)$ only in the last component, namely if $y \in R(T)$ $x = \sum_{i=1}^{\infty} \alpha_i e_i$, and $T(x) = y$, then

$$\langle T_n(x_n), e_n \rangle = u_n \alpha_{n-1} + d_n \alpha_n$$

and

$$\langle T(x), e_n \rangle = u_n \alpha_{n-1} + d_n \alpha_n + c_n \alpha_{n+1}.$$

Thus $T_n(x_n) + \alpha_{n+1}c_n e_n = y_n$ which implies that $T_n^{-1}(y_n) = x_n + c_n \alpha_{n+1} T_n^{-1}(e_n)$. As $n \rightarrow \infty$, $x_n \rightarrow x$, $\alpha_n \rightarrow 0$ and $\{T_n^{-1}(e_n)\}$ is bounded, we see that $T_n^{-1}(y_n) \rightarrow x$. This also implies that T is 1-1 (by taking $y = 0$).

To prove that T is onto, take a $y \in H$. Then $y = \sum_{i=1}^{\infty} \beta_i e_i, y_n = \sum_{i=1}^n \beta_i e_i$. As T_n is onto $\exists x^n \in H_n$ such that $T_n x^n = y_n$. Further

$$y = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} T_n(x^n). \quad (2.1)$$

If we write

$$x^n = \alpha_1^n e_1 + \alpha_2^n e_2 + \cdots + \alpha_n^n e_n,$$

then one can show that

$$T(x^n) - T_n(x^n) = \alpha_n^n u_{n+1} e_{n+1} \quad (2.2)$$

and $\alpha_n^n = \sum_{i=1}^n \beta_i < e_i, (T_n)^{*^{-1}} e_n >$.

But making use of the fact that $\{T_n^{*-1} e_n\}$ is bounded, we prove that $\alpha_n^n u_{n+1} \rightarrow 0$ as $n \rightarrow \infty$. Similarly writing

$$x^{n+1} - x^n = (\beta_{n+1} - \alpha_n^n u_{n+1}) T_{n+1}^{-1}(e_{n+1})$$

and

$$T_n^{-1}(e_n) = (-1)^{n+1} (\det T_n)^{-1} [c_1 c_2 \cdots c_{n-1} e_1 - c_2 \cdots c_{n-1} (\det T_1) e_2 + \cdots + (-1)^{n-1} \det T_{n-1} e_n]$$

using the fact that $\{T_n^{-1} e_n\}$ is bounded, we can show that $\{x_n\}$ is a Cauchy sequence is H . Let $x = \lim_{n \rightarrow \infty} x_n$. Since $\alpha_n^n u_{n+1} \rightarrow 0$ as $n \rightarrow \infty$, from (2.1) and (2.2), it follows that T is onto. Complete proof is given in [4].

The following corollary gives a verifiable criterion for an infinite matrix to satisfy the conditions in the above theorem

Corollary 2.2. With all notations as in Theorem 2.1 suppose the following condition hold: There exists η_0, η, η' such that $\eta_0 > 0, 0 \leq \eta, \eta' < 1$ and

- (i) $|d_n| \geq \eta_0$ for all n
- (ii) $|d_n| |d_{n+1}| \geq 4 |u_{n+1}| |c_n|$ for all n
- (iii) $|d_n| |d_{n+1}| \geq \frac{4}{\eta} |u_{n+1}| |u_n|$ for all n
- (iv) $|d_n| |d_{n+1}| \geq \frac{4}{\eta'} |c_n| |c_{n-1}|$ for all n

Then each T_n is invertible and $\{\|T_n^{-1}(e_n)\|\}, \{\|T_n^{*-1}(e_n)\|\}$ are both bounded sequences. Hence the conclusions of Theorem 2.1 hold.

Proof. Detailed proof is given in [4].

Example 2.3 Let $u_n = c_n = 1$ for all n are define d_n by the following:

for $n = 0, 1, \dots$

$$d_{2n+1} = 10 - \frac{1}{n}$$

$$d_{2n+2} = \frac{5}{d_{2n+1}}$$

It is easy to see that the condition in Corollary 2.2 are satisfied, by taking

$$\eta_0 = \frac{1}{2}, \quad \eta = \eta' = \frac{4}{5}$$

3. Algorithm

In this section, we present an algorithm to compute the solution $x = (\alpha_1, \dots)$ when the sequences $\{u_n\}, \{c_n\}, \{d_n\}, \{\beta_n\}$ are given and apply this algorithm to Example 2.3.

Before presenting the algorithm, let us build up the essential requirements. For convenience, we assume that $\{u_n\}, \{c_n\}, \{d_n\}, \{\beta_n\}$ are all real. We have

$$\begin{aligned} \|T_{n+1}^{-1}(e_{n+1})\|^2 &= \frac{1}{(\det T_{n+1})^2} (c_1^2 \dots c_n^2 + c_2^2 \dots c_n^2 \det T_1^2 + \dots + c_n^2 \det T_{n-1}^2 + \det T_n^2) \\ &= (\det T_{n+1})^{-2} (c_n^2 \det T_n^2 \|T_n^{-1} e_n\|^2 + \det T_n^2) \\ &= \det T_n^2 \det T_{n+1}^{-2} (1 + c_n^2 \|T_n^{-1} e_n\|^2). \end{aligned}$$

Similarly,

$$\|T_{n+1}^{*-1} e_{n+1}\|^2 = \det T_n^2 \det T_{n+1}^{-2} (1 + u_{n+1}^2 \|T_n^{*-1} e_n\|^2).$$

Also, $\|T_1^{-1} e_1\|^2 = \|T_1^{*-1} e_1\|^2 = d_1^{-2}$. Further $\det T_n$ satisfies the recurrence relation:

$$\det T_n = d_n \det T_{n-1} - u_n c_{n-1} \det T_{n-2} \quad n \geq 2$$

with $\det T_0 = 1$ and $\det T_1 = d_1$. As proved in theorem 2.1, we also have

$$x^{n+1} - x^n = (\beta_{n+1} - \alpha_n^n u_{n+1}) T_{n+1}^{-1}(e_{n+1})$$

where $x^n = \alpha_1^n e_1 + \alpha_2^n e_2 + \dots + \alpha_n^n e_n$. We write $x^{n+1} = \alpha_1^{n+1} e_1 + \alpha_2^{n+1} e_2 + \dots + \alpha_{n+1}^{n+1} e_{n+1}$ and express $T_{n+1}^{-1}(e_{n+1})$ in terms of $T_n^{-1}(e_n)$. Then by equating the coefficients of e_{n+1} in $x^{n+1} - x^n$, we get

$$\alpha_{n+1}^{n+1} = (\det T_n)(\det T_{n+1})^{-1} (\beta_{n+1} - \alpha_n^n u_{n+1}).$$

For the sake of convenience let us use the following notations. $\beta_n = b_n$, $\alpha_n^n = a_n$, $\det T_n = t_n$. $\|T_n^{-1} e_n\|^2 = r_n$, $\|T_n^{*-1} e_n\|^2 = s_n$.

Algorithm. Let M_1, M_2 denote very large numbers and $\epsilon > 0$ a very small number.

Step 1. Fix n . Input d_1, d_2, \dots, d_n , c_1, c_2, \dots, c_{n-1} , u_2, u_3, \dots, u_n , b_1, b_2, \dots, b_n . Define $t_0 = 1$, $t_1 = d_1$, $r_1 = \frac{1}{d_1^2}$, $s_1 = \frac{1}{d_1^2}$, $a_1 = \frac{b_1}{d_1}$, $x_1 = \frac{b_1}{d_1} e_1$, $T_1^{-1}(e_1) = \frac{1}{d_1} e_1$. For $i = 2, 3, \dots, n$ we compute the following.

Step 2. Let $t_i = d_i(t_{i-1}) - u_i c_{i-1} t_{i-2}$. If $t_i = 0$ for some i we stop further computation and print "algorithm will not apply as T_i is not invertible." Otherwise we proceed to Step 3.

Step 3. Let $r_i = (1 + c_{i-1}^2 r_{i-1}) t_{i-1}^2 / t_i^2$. Again if $r_i > M_1$ for some i , then stop further computations and print "algorithm will not apply as r_i is large." Otherwise we proceed to Step 4.

Step 4. Let $s_i = (1 + u_i s_{i-1}^2) t_{i-1}^2 / t_i^2$. If $s_i > M_2$ for some i then stop further processing and print "algorithm will not apply as s_i value is large." Otherwise proceed to Step 5.

Step 5. Let $a_i = (b_i - a_{i-1} u_i) t_{i-1} / t_i$, $T_i^{-1}(e_i) = t_{i-1} / t_i \{-c_{i-1} T_{i-1}^{-1}(e_{i-1}) + e_i\}$, $x_i = x_{i-1} + (b_i - a_{i-1} u_i) T_i^{-1}(e_i)$.

Step 6. Calculate the error $err(n) = |a_{n-1}u_n|$ and the relative error $rerr(n) = err(n)/\|x\|$. If $rerr(n) < \epsilon$ we stop and print x_n and $rerr(n)$. Otherwise repeat Step1 to Step 5 until $rerr(n) < \epsilon$.

We applied this algorithm to the operator in Example 2.3 and by taking the right hand side $b_n = \frac{1}{n}$ for all n . Results obtained (using MATLAB) are tabulated for the following values of n .

Number of iterations n	Relative error rerr(n)
20	0.02230
30	0.11160
40	0.00740
50	0.00520
100	0.00180
150	0.00009

After 150 iterations, the relative error becomes negligible.

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