MR 2319017, Math Today (Ahmedabad) 19, 2001 C(1), 3-8.

SPECTRAL REPRESENTATION OF A QUATERNIONIC HILBERT SPACE

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ABSTRACT

Let H be a quaternionic Hilbert space and $B(H)$ denote the set of all bounded real-linear operators on H. Let A be a closed $*$-subalgebra of $B(H)$ such that every element in A is normal and $qI \in A$ for all $q \in \mathcal{H}$, the set of all real quaternions. It is shown that every operator in A is unitarily equivalent to a direct sum of left multiplications on some L^2 spaces of quaternion-valued measurable functions.

Key Words: quaternionic Hilbert space, real Banach algebra,normal operators, spectral representation.

1991 Mathematics subject classification; Primary 46L05

1 Introduction

The possibility of replacing the complex numbers by the quaternions in Hilbert space theory was studied by Teichmuller, Birkhoff and Von Neumann in early thirties. These Hilbert spaces are found to be useful in quantum mechanics, in particular to quantum field theories and nonabelian gauge fields ([1], [5]). Since then the spectral theory and the theory of group representations in quaternionic Hilbert spaces received greater attention. If H is a quaternionic Hilbert space, with inner product $(.,.)$, then H is also a real Hilbert space with inner product given by $< .,. >= Re(.,.)$ and the (real) algebra of all bounded (quaternion) linear operators on H is a subalgebra of the algebra of all bounded real linear operators on this underlying real Hilbert space. Teichmuller [4], Viswanath [6] and others developed the spectral theory for quaternion linear bounded normal operators. However the class of all quaternion-linear operators is a very restricted class. Even a quaternion multiple of the identity operator is not quaternion linear; it is only real linear. Hence it is considered worthwhile to deal with real linear operators and algebra of such operators. Powers [4] has developed spectral theory for bounded real linear normal operators on a quaternionic Hilbert space. In this paper we use real Banach algebra techniques in studying the spectral representation of bounded real linear operators on a quaternionic Hilbert space. In particular we use the recently developed analogues of the classical Gelfand–Naimark theorems ([2], [3]).

2 Preliminaries

Let $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{H}$ denote the sets of real numbers, complex numbers, and real quaternions respectively.

For $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathcal{H}$;
q^* is defined as
$q^* = q_0 - q_1 i - q_2 j - q_3 k$ and
$|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$.

Note that $|q|^2 = q^*q = qq^*$.

Definition 1: A quaternionic Hilbert space H is a left module over $\mathcal{H}$ with $\mathcal{H}$ valued inner product which is complete in the norm induced by the inner product (See [3] for a complete list of axioms).

Example 2: Let (X, m, μ) be a measure space with μ, a positive finite measure. Let $L^2_{\mathcal{H}}(X, \mu)$ be the set of all equivalence classes of quaternion–valued measurable functions f on X such that $\int_X |f|^2 d\mu < \infty$.

For $f,\ g \in L^2_{\mathcal{H}}(X, \mu)$ define the inner product by
$(f, g) = \int_X fg^* d\mu$,
where $g^*(\lambda) = (g(\lambda))^*$, for all $\lambda \in X$.

Then with respect to this inner product, $L^2_{\mathcal{H}}(X, \mu)$ is a quaternionic Hilbert space ([4], [6]).

For any quaternion valued essentially bounded measurable function h on X, the operator R_h defined by $R_h(f) = fh$ (right multiplication by h) is a quaternion linear bounded normal operator on $L^2_{\mathcal{H}}(X, \mu)$. Viswanath [6] has proved that every quaternion linear normal operator on a quaternionic Hilbert space is built up of operators of the form R_h. The operator L_h is defined by $L_h(f) = hf$ (left multiplication by h on $L^2_{\mathcal{H}}(X, \mu)$)is a bounded real–linear operator but not a quaternion linear operator and the adjoint L^*_h of this operator is L_{h^*}, the left multiplication by h^*. Recall that $< .,. >= Re(.,.)$ is a real inner product on $L^2_{\mathcal{H}}(X, \mu)$. Thus,

$$\begin{aligned} < L_h f, g > &= Re \int_X hfg^* d\mu \\ &= \int_X Re\ hfg^* d\mu \\ &= \int_X Re\ fg^* h d\mu \\ &= Re \int_X f(h^*g)^* d\mu \\ &= < f, h^*g >,\ for\ all\ f,\ g\ \in L^2_{\mathcal{H}}(X, \mu) \end{aligned}$$

It may be noted that L_{h^*} is the adjoint of L_h with respect to the real inner product

$< .,. >$, and not with respect to the quaternionic inner product $(.,.)$. We shall prove that every real linear normal operator on a quaternionic Hilbert space is built up from operators of this form. We shall need the following results from ([2], [3]) to prove the main theorem.

Remark 3: Let A be a real C^*-algebra in which every element is normal and X be the set of all non-zero real algebra homomorphisms of A into $\mathcal{H}$. For a in A, define a map $\hat{a} : X \to \mathcal{H}$ by $\hat{a}(\pi) = \pi(a)$, for all $\pi \in X$.

Let $\hat{A} = \{\hat{a} : a \in A\}$. Then with respect to the weak $\hat{A}$ topology (i.e. the weakest topology with respect to which $\hat{a}$ is continuous for each $a \in A$). X is a nonempty compact Hausdorff space. Furthermore, the map $a \to \hat{a}$ is an isometric $*$-isomorphism of A into $C(X, \mathcal{H})$ ([2], theorem 3), where $C(X, \mathcal{H})$ is the real Banach algebra of all continuous $\mathcal{H}$-valued functions on the compact Hausdorff space X. In addition, if A is a module over $\mathcal{H}$, then all the quaternion multiples of the identity belong to A. In this case, the map $a \to \hat{a}$ is onto ([3], Corollary 6), that is $\hat{A} = C(X, \mathcal{H})$.

3 The spectral representation

Theorem 4: Let H be a quaternionic Hilbert space with the inner product $(.,.)$ and A be a closed real $*$-algebra of bounded real-linear operators on H such that every element in A is normal and $qI \in A$, for all $q \in \mathcal{H}$. Suppose there exists $x \in H$ such that $\{Tx : T \in A\}$ is dense in H. Then there exists a compact Hausdorff space X, a positive regular Borel measure μ on X and a real-linear onto isometry $U : H \to L^2_{\mathcal{H}}(X, \mu)$ such that for all $S \in A$, USU^{-1} is a left multiplication operator on $L^2_{\mathcal{H}}(X, \mu)$.

Proof:

Let $X = \{\pi : A \to \mathcal{H}\}$, where π is a nonzero real algebra homomorphism.

Then by Remark 3, X is a nonempty compact Hausdorff space and the mapping $T \to \hat{T}$ is an isometric $*$-isomorphism of A onto $C(X, \mathcal{H})$.

Define $\psi : C(X, \mathcal{H}) \to \mathcal{R}$ by $\psi(\hat{T}) =< Tx, x >= Re(Tx, x)$ for $T \in A$. Since $T \to \hat{T}$ is a $*$-isomorphism of A onto $C(X, \mathcal{H})$, $T^* = T$ if and only if $\hat{T} \in C_{\mathcal{R}}(X)$, the algebra of all continuous real-valued functions on X.

Now, $\psi/C_{\mathcal{R}}(X)$ is a bounded real linear functional from $C_{\mathcal{R}}(X)$ to $\mathcal{R}$. By the Riesz representation theorem, there exists a unique regular real Borel measure μ on X such that $< Tx, x >= \psi(\hat{T}) = \int_X \hat{T} d\mu$, for all self-adjoint $T \in A$ and $\| \psi/C_{\mathcal{R}}(X) \| = \| \mu \| \leq \| x \|^2$.

If $\hat{T}$ is positive, then $\hat{T} = f^2$ for some non-negative $f \in C_{\mathcal{R}}(X)$ and $f = \hat{S}$ for some self-adjoint $S \in A$. Now, $< Tx, x > = < S^2 x, x > = \| Sx \|^2 \geq 0$. Thus $\psi/C_{\mathcal{R}}(X)$ is positive. So the measure μ is positive.

Define $U : H \longrightarrow L^2_{\mathcal{H}}(X, \mu)$ by $U(Tx) = \hat{T}$, for all $T \in A$. Since T^*T is self-adjoint,

$$\begin{aligned} \| Tx \|^2 &= < T^*Tx, x >= \psi(\hat{T^*T}) \\ &= \psi(| \hat{T} |^2) = \int_X | \hat{T} |^2 d\mu \end{aligned}$$

This shows that U is a real linear isometry from a dense subset of H into $L^2_{\mathcal{H}}(X, \mu)$. Hence, U can be extended uniquely to H. Clearly, this extension is also a real-linear isometry. We shall denote this extension also by U. Since range of U contains $C(X, \mathcal{H})$, which is dense in $L^2_{\mathcal{H}}(X, \mu)$, U is onto.

Now, let $S \in A$. Then for $T \in A$, that is for $\hat{T} \in C(X, \mathcal{H})$,

$U\,S\,U^{-1}(\hat{T}) = U(STx) = \hat{ST} = \hat{S}\hat{T} = L_{\hat{S}}\hat{T}$.

Thus, $U\,S\,U^{-1} = L_{\hat{S}}$ on $C(X, \mathcal{H})$.

Since $C(X, \mathcal{H})$ is dense in $L^2_{\mathcal{H}}(X, \mu)$, $U\,S\,U^{-1} = L_{\hat{S}}$ on $L^2_{\mathcal{H}}(X, \mu)$.

Remark 5: If there does not exist a vector $x \in H$ such that $\{Tx : T \in A\}$ is dense in H, then by applying Zorn's lemma we get $H = \sum_\alpha H_\alpha$, the orthogonal direct sum of closed subspaces H_α such that each H_α contains a vector x_α satisfying $\{Tx_\alpha : T \in A\}$ is dense in H_α and each H_α is invariant under T and T^* for all $T \in A$. Then by applying the above Theorem 4 to each H_α, we can obtain a spectral representation of H onto a direct sum $\sum_\alpha L^2_{\mathcal{H}}(X, \mu_\alpha)$ relative to A.

Corollary 6: Let A be as in Theorem 4. The following statements are equivalent for $S \in A$.

(i) $S^* = S$

(ii) USU^{-1} is a left multiplication by a real-valued function.

(iii) $ST = TS$, for all $T \in A$

Proof:

(i) implies (ii)

From the above theorem, USU^{-1} is a left multiplication by $\hat{S}$, which is a real-valued function, if $S^* = S$.

(ii) implies (iii)

If $\hat{S}$ is real-valued, then $\hat{S}\hat{T} = \hat{T}\hat{S}$, for all $T \in A$. Therefore, $\hat{ST} = \hat{TS}$.

Hence, $ST = TS$, as $T \to \hat{T}$ is a bijection.

(iii) implies (i)

If $ST = TS$ for all $T \in A$, then $\hat{S}\hat{T} = \hat{T}\hat{S}$, for all $T \in A$.

Let $\hat{S} = f_0 + f_1 i + f_2 j + f_3 k$ for some real-valued functions f_0, f_1, f_2, f_3.

Since $\hat{A} = C(X, \mathcal{H})$, the constant function $i = \hat{T}$, for some $T \in A$.

Therefore, $(f_0 + f_1 i + f_2 j + f_3 k)i = i(f_0 + f_1 i + f_2 j + f_3 k)$

That is, $f_0 i - f_1 - f_2 k + f_3 j = f_0 i - f_1 + f_2 k - f_3 j$.

This implies that $f_2 = 0 = f_3$.

Similarly, if $\hat{T}$ is a constant function j, for some $T \in A$, we have, $f_1 = 0 = f_3$. Therefore, $\hat{S} = f_0$, a real-valued function.

This implies $S^* = S$.

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